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A CR twistor space of a G_2-manifold

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ABSTRACT

Let M be a G_2-manifold. We consider an almost CR-structure on the sphere bundle of unit tangent vectors on M, called *the CR twistor space*. This CR-structure is integrable if and only if M is a holonomy G_2-manifold. We interpret G_2-instanton bundles as CR-holomorphic bundles on its twistor space.

1. Introduction

1.1. A CR twistor space of a G_2-manifold

This note is inspired by the Claude LeBrun's paper [12]. In his work, LeBrun constructed a twistor space for a 3-manifold, which happens to be a CR-manifold of real dimension 5. The geometry of G_2-manifolds is remarkably similar to the geometry of 3-manifolds, and it is not surprising that an analogue of LeBrun's construction can be obtained.

Definition 1.1. Let M be a smooth manifold, $B \subset TM$ a subbundle in its tangent bundle, and $I \in \operatorname{End} B$ its automorphism, $I^2 = -\operatorname{Id}_B$. Consider the $(1,0)$ and $(0,1)$-bundles $B^{1,0}, B^{0,1} \subset B \otimes \mathbb{C}$, which are the eigenspaces of I corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$. The subbundle $B^{1,0} \subset TM \otimes \mathbb{C}$ is called **a CR-structure on** M if it is *involutive*, that is, satisfies $[B^{1,0}, B^{1,0}] \subset B^{1,0}$.

Example 1.2. If $B = TM$, CR-structures are the same as complex structures. For any codimension 1 real submanifold M in a complex manifold (X, I), the intersection $B := TX \cap I(TX)$ is a complex subbundle of codimension 1 in TX, and the restriction $I|_B$ defines a CR-structure.

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G_2-manifolds originally appeared in Berger's classification of holonomy [1,2]. The first examples of G_2-manifolds were obtained by R. Bryant and S. Salamon [3]. The compact examples of G_2-manifolds were constructed by D. Joyce [9,10]. In this introduction we follow the approach to G_2-geometry which is due to N. Hitchin (see [8]).

Definition 1.3. Let $\rho \in \Lambda^2\mathbb{R}^7$ be a 3-form on $\mathbb{R}^7$. We say that ρ is **non-degenerate** if the dimension of its stabilizer is maximal:

$$\dim St_{GL(7)}\rho = \dim GL(7) - \dim \Lambda^3\big(\mathbb{R}^7\big) = 49 - 35 = 14.$$

In this case, $St(\rho)$ is one of two real forms of a 14-dimensional Lie group $G_2(\mathbb{C})$. We say that ρ is **non-split** if it satisfies $St(\rho|_x) \cong G_2$, where G_2 denotes the compact real form of $G_2(\mathbb{C})$. **A G_2-structure** on a 7-manifold is a 3-form $\rho \in \Lambda^3(M)$, which is non-degenerate and non-split at each point $x \in M$. We shall always consider a G_2-manifold as a Riemannian manifold, with the Riemannian structure induced by the G_2-structure as indicated below.

Remark 1.4. Such a form defines a $\Lambda^7 M$-valued metric on M:

$$g(x, y) = (\rho \rfloor x) \wedge (\rho \rfloor y) \wedge \rho \tag{1.1}$$

(we denote by $\rho \rfloor x$ the contraction of ρ with a vector field x). The Riemannian volume form associated with this metric gives a section of $\Lambda^7 M \otimes (\Lambda^7 M)^{7/2}$. Squaring and taking the 9-th degree root, we obtain a trivialization of the volume. Then (1.1) defines a metric g on M, by construction G_2-invariant.

Definition 1.5. A G_2-structure is called **an integrable G_2-structure**, if ρ is preserved by the corresponding Levi-Civita connection. An integrable G_2-manifold is a manifold equipped with an integrable G_2-structure. Holonomy group of such a manifold clearly lies in G_2; for this reason, the integrable G_2-manifolds are often called **holonomy G_2-manifolds**.

Remark 1.6. In the literature, "the G_2-manifold" often means a "holonomy G_2-manifold", and "G_2-structure" "an integrable G_2-structure". A G_2-structure which is not necessarily integrable is called "an almost G_2-structure", taking analogy from almost complex structures.

Remark 1.7. As shown in [7], integrability of a G_2-structure induced by a 3-form ρ is equivalent to $d\rho = d(*\rho) = 0$. For this reason the 4-form $*\rho$ is called **a fundamental 4-form of a G_2-manifold**, and ρ **the fundamental 3-form**.

Remark 1.8. Let $V = \mathbb{R}^7$ be a 7-dimensional real space equipped with a non-degenerate 3-form ρ with $St_{GL(7)}(\rho) = G_2$. As in Remark 1.4, one can easily see that V has a natural G_2-invariant metric. For each vector $x \in V$, $|x| = 1$, its stabilizer $St_{G_2}(x)$ in G_2 is isomorphic to $SU(3)$. Indeed, the orthogonal complement $x^\perp$ is equipped with a symplectic form $\rho \rfloor x$, which gives a complex structure $g^{-1} \circ (\rho \rfloor x)$ as usual. This gives an embedding $St_{G_2}(x) \hookrightarrow U(3)$. Since the space of such x is S^6, and the action of G_2 in S^6 is transitive, one has $\dim St_{G_2}(x) = \dim G_2 - \dim S^6 = 8 = \dim U(3) - 1$. To see that $St_{G_2}(x) = SU(3) \subset U(3)$ and not some other codimension 1 subgroup, one should notice that $St_{G_2}(x)$ preserves two 3-forms $\rho|_{x^\perp}$ and $\rho^* \rfloor x|_{x^\perp}$, where $\rho^* = *\rho$ is the fundamental 4-form of V. A simple linear-algebraic calculation implies that $\rho|_{x^\perp} + \sqrt{-1}\rho^* \rfloor x|_{x^\perp}$ is a holomorphic volume form on $x^\perp$, which is clearly preserved by $St_{G_2}(x)$. Therefore, the natural embedding $St_{G_2}(x) \hookrightarrow U(3)$ lands $St_{G_2}(x)$ to $SU(3)$. Using the dimension count $\dim St_{G_2}(x) = \dim SU(3)$ (see above), we show that the embedding $St_{G_2}(x) \hookrightarrow SU(3)$ is also surjective.

Let now M be an almost G_2-manifold. From Remark 1.8 it follows that with every vector $x \in TM$, $|x| = 1$, one can associate a complex Hermitian structure on its orthogonal complement $x^\perp$. The easiest way to define this structure is to notice that $x^\perp$ is equipped with a symplectic structure $\rho \rfloor x$ and a metric $g|_{x^\perp}$, which can be considered as a real and imaginary parts of a complex-valued semilinear Hermitian product. Then the complex structure is obtained as usual, as $I := (\rho \rfloor x) \circ g^{-1}$.

Definition 1.9. Consider now the unit sphere bundle S^6M over M, with the fiber S^6, and let $T_{\mathrm{hor}} S^6 M \subset TS^6M$ be the horizontal subbundle corresponding to the Levi-Civita connection. This subbundle has a natural section θ; at each point $(x, m) \in S^6M$, $m \in M$, $x \in T_m M$, $|x| = 1$, we take $\theta|_{(x,m)} = x$, using the standard identification $T_{\mathrm{hor}} S^6 M|_{(x,m)} = T_m M$. Denote by $B \subset T_{\mathrm{hor}} S^6 M$ the orthogonal complement to θ in $T_{\mathrm{hor}} S^6 M$. Since at each point $(x, m) \in S^6M$, the restriction $B|_{(x,m)}$ is identified with $x^\perp \subset T_m M$, this bundle is equipped with a natural complex structure, that is, an operator $I \in \operatorname{End} B$, $I^2 = -\operatorname{Id}_B$.

The main result of the present paper is the following theorem.

Theorem 1.10. *Let M be an almost G_2-manifold, $S^6M \subset TM$ its unit sphere bundle, and $B \subset TS^6M$ a subbundle of its tangent bundle constructed above, and equipped with the complex structure I as above. Then $B^{0,1} \subset B \otimes \mathbb{C} \subset TS^6M \otimes \mathbb{C}$ is involutive if and only if M is a holonomy G_2-manifold.*

We prove Theorem 1.10 in Section 4.

Definition 1.11. Let M be a holonomy G_2-manifold, and

$$\mathrm{Tw}(M) := \left(S^6 M, B, I\right)$$

the CR-manifold constructed in Theorem 1.10. Then $\mathrm{Tw}(M)$ is called **a CR-twistor space of** M.

1.2. Applications of twistor geometry

G_2 instanton bundles were introduced in [6], and much studied since then. This notion is a special case of a more general notion of an instanton on a calibrated manifold, which is already well developed. Many estimates known for 4-dimensional manifolds (such as Uhlenbeck's compactness theorem) can be generalized to the calibrated case [15,14].

Recently, G_2-instantons became a focus of much activity because of attempts to construct a higher-dimensional topological quantum field theory, associated with G_2 and 3-dimensional Calabi–Yau manifolds [5].

Definition 1.12. Let M be a G_2-manifold, and $\Lambda^2 M = \Lambda^2_7(M) \oplus \Lambda^2_{14}(M)$ the irreducible decomposition of the bundle of 2-forms $\Lambda^2(M)$ associated with the G_2-action (Section 3.2). A vector bundle (B, ∇) with connection is called **a G_2-instanton** if its curvature lies in $\Lambda^2_{14}(M) \otimes \mathrm{End}(B)$.

Remark 1.13. The instanton connection is the same as a self-dual connection in the context of G_2-geometry. In particular, it minimizes the curvature functional, in the same way as the usual instantons minimize the curvature functional. Therefore, the instanton connections are solutions of the Yang–Mills equation (see [15]).

Remark 1.14. A tangent bundle and all its tensor powers are obviously G_2-instantons (see Step 1 in the proof of Theorem 1.10 in Section 4).

The definition of CR-holomorphic bundles is a straightforward generalization of a usual differential-geometric notion of a holomorphic bundle as a bundle equipped with a Dolbeault differential $\bar\partial$ which satisfies $\bar\partial^2 = 0$.

Definition 1.15. Let $(M, B, I \in \mathrm{End}\, B)$ be a CR-manifold, and $B^{0,1} \subset B \otimes \mathbb{C}$ the $\sqrt{-1}$-eigenspace of B. Using the Cartan's formula, we define the CR Dolbeault differential

$$\Lambda^k\left(B^{0,1}\right)^* \xrightarrow{\bar\partial_B} \Lambda^{k+1}\left(B^{0,1}\right)^*$$

as usual,

$$\begin{aligned}(\bar\partial_B \alpha)(b_1, \dots, b_{k+1}) := {}& \sum_i (-1)^i \operatorname{Lie}_{b_i} \alpha(b_1, \dots, \check{b}_i, \dots, b_{k+1}) \\ &+ \sum_{i<j} (-1)^{i+j+1} \alpha\left([b_i, b_j], b_1, \dots, \check{b}_i, \dots, \check{b}_j, \dots, b_{k+1}\right).\end{aligned}$$

Let E be a vector bundle, and $\bar\partial_E : E \longrightarrow E \otimes (B^{0,1})^*$ an operator which satisfies the Leibniz rule

$$\bar\partial_E(f\xi) = \xi \otimes \bar\partial_B f + f \bar\partial_E \xi.$$

We extend $\bar\partial_E$ to

$$E \otimes \Lambda^k\left(B^{0,1}\right)^* \xrightarrow{\bar\partial_E} E \otimes \Lambda^{k+1}\left(B^{0,1}\right)^*$$

using the same Leibniz formula,

$$\bar\partial_E(\xi \otimes \alpha) = \xi \otimes \bar\partial_B \alpha + \bar\partial_E \xi \otimes \alpha$$

for any $\xi \in E, \alpha \in \Lambda^k(B^{0,1})^*$. A bundle $(E, \bar\partial_E)$ is called **CR-holomorphic**, if $\bar\partial_E^2 = 0$. In this case, $\bar\partial_E$ is called **an operator of CR-holomorphic structure**.

Definition 1.16. Let $(M, B, I \in \mathrm{End}\, B)$ be a CR-manifold, and (E, ∇) a complex vector bundle on M with connection. Denote by

$$\Pi : \Lambda^i M \longrightarrow \Lambda^i\left(B^{0,1}\right)^*$$

the restriction map, and let

$$\bar{\partial}_E := \nabla \circ \Pi : E \longrightarrow E \otimes \left(B^{0,1}\right)^*$$

be the $B^{0,1}$-part of the connection. We say that (E, ∇) is **CR-holomorphic** if $\bar{\partial}_E$ is an operator of CR-holomorphic structure on E.

Theorem 1.17. *Let M be a holonomy G_2-manifold, (E, ∇) a bundle with connection, and $\mathrm{Tw}(M) \xrightarrow{\pi} M$ its twistor space defined as in Definition 1.11. Then the following assertions are equivalent:*

(i) *The pullback $(\pi^*E, \pi^*\nabla)$ is a CR-holomorphic bundle on $\mathrm{Tw}(M)$.*
(ii) *(E, ∇) is a G_2-instanton.*

Proof. Let $\nabla^2 R \in \Lambda^2 M \otimes \operatorname{End} E$ be the curvature form of E, π^*R the curvature of $(\pi^*E, \pi^*\nabla)$, and $\Pi(\pi^*R) \in \Lambda^2(B^{0,1})^* \otimes \operatorname{End} E$ its restriction to $B^{0,1}$. Clearly, $\Pi(\pi^*R) = 0$ if and only if $(\pi^*E, \pi^*\nabla)$ is CR-holomorphic. However, $\Pi(\pi^*R) = 0$ if and only if π^*R is of Hodge type $(1,1)$ on B, and this is equivalent to $R \in \Lambda^2_{14} \otimes \operatorname{End} E$, as follows from Proposition 3.2. $\square$

Remark 1.18. Following the same approach as in the paper [13], it is possible to deduce from Theorem 1.10 that the space of knots (non-parametrized loops) in a holonomy G_2-manifold is formally Kaehler (see the forthcoming paper [16]). For 3-manifolds this theorem is due to J.-L. Brylinski [4].

2. Differential forms on $\mathrm{Tot}(\Lambda^k M)$

In this section, we extend the standard results about the Hamiltonian 2-form on T^*M to the total space $\mathrm{Tot}(\Lambda^k(M))$ of the bundle of k-forms. These generalizations are elementary, but used further on in this paper.

Let M be a smooth manifold, and $X = \mathrm{Tot}(\Lambda^k M)$ a total space of a bundle of k-forms. On X, a pair of forms is defined: a k-form Θ, which is an analogue of the 1-form $\sum q_i\, dp_i$ on T^*M, and a $(k+1)$-form $\Xi = d\Theta$, which is an analogue of the symplectic form.

Let $\pi : X \longrightarrow M$ be a standard projection. At a point $(\lambda, m) \in X$, with $m \in M$ and $\lambda \in \Lambda^k(M)|_m$, we define

$$\Theta(x_1, \dots, x_k) := \lambda\bigl(D\pi(x_1), D\pi(x_2), \dots, D\pi(x_k)\bigr), \tag{2.1}$$

where $D\pi : TX \longrightarrow TM$ is the differential of π.

Let $\Xi := d\Theta$. In local coordinates $p_1, \dots, p_n$ on M,

$$q_{i_1,\dots,i_k} := dp_{i_1} \wedge dp_{i_2} \wedge \dots \wedge dp_{i_k}$$

on the fibers of π, Ξ can be written as sum

$$\Xi = \sum_{i_1 < i_2 < \dots < i_k} dq_{i_1,\dots,i_k} \wedge dp_{i_1} \wedge dp_{i_2} \wedge \dots \wedge dp_{i_k} \tag{2.2}$$

(this is clear from the same argument as used to obtain a more familiar relation $d(\sum q_i\, dp_i) = \sum dq_i \wedge dp_i$). Further on, we shall need the following elementary lemma.

Lemma 2.1. *Let M be a manifold equipped with a torsion-free connection ∇, $X = \mathrm{Tot}(\Lambda^k M)$ the total space of the bundle of k-forms, and $\Xi \in \Lambda^{k+1}(X)$ the fundamental $(k+1)$-form constructed above. Using the connection ∇, we split the tangent bundle onto horizontal and vertical components, $TX = T_{\mathrm{hor}} X \oplus T_{\mathrm{vert}} X$. Then $\Xi|_{T_{\mathrm{hor}} X} = 0$.*

Proof. Choose coordinates $p_1, \dots, p_n$ on M, and let $p_1, \dots, p_n$, $q_{i_1,\dots,i_k}$, $1 \leqslant i_1 < i_2 < \dots < i_k \leqslant n$ be the coordinates on X constructed as above. This coordinate system induces a flat torsion-free connection ∇_0 on TM and $\Lambda^k M$. For an arbitrary connection ∇ on TM, one obtains a splitting $TX = T_{\mathrm{vert}} \oplus T_{\mathrm{hor}}(\nabla)$. Write $\nabla = \nabla_0 + A$, where $A \in \Lambda^1 M \otimes \operatorname{End}(TM)$ is an $\operatorname{End}(TM)$-valued 1-form on M. The difference between $T_{\mathrm{hor}}(\nabla)$ and $T_{\mathrm{hor}}(\nabla_0)$ can be expressed as a section of $\operatorname{Hom}(\pi^*TM, T_{\mathrm{vert}} X)$, at (λ, m) giving $A_k(\lambda) \in \Lambda^1(M) \otimes T_{\mathrm{vert}} X$, where $A_k \in \Lambda^1(M) \otimes \operatorname{End}(\Lambda^k M)$ is the connection form on $\Lambda^k(M)$ induced from ∇, $A_k = \nabla - \nabla_0$. Denote $A_k(\lambda)$ by $E : \pi^*\Lambda^1 M \longrightarrow T_{\mathrm{vert}} X = \pi^*\Lambda^k M$. Using the formula for Ξ written in coordinates as in (2.2), we obtain

$$\Xi\bigl(v'_{i_1}, v'_{i_2}, \dots, v'_{i_{k+1}}\bigr) = \Xi\bigl(v_{i_1} + E(v_{i_1}), v_{i_2} + E(v_{i_2}), \dots, v_{i_{k+1}} + E(v_{i_{k+1}})\bigr), \tag{2.3}$$

where $v_i := \frac{d}{dp_i}$ are coordinate vector fields on X, and $v'_i = v_i + E(v_i)$ the corresponding sections of $T_{\mathrm{hor}}(\nabla)$. As follows from (2.2), the latter expression gives

$$\Xi\left(v'_{i_1}, v'_{i_2}, \dots, v'_{i_{k+1}}\right) = \frac{1}{(k+1)!}\sum (-1)^{|\sigma|} E(v_{\sigma(i_1)})(v_{\sigma(i_2)}, v_{\sigma(i_3)}, \dots, v_{\sigma(i_{k+1})}) \tag{2.4}$$

where the sum is taken over all permutations $\sigma \in S_{k+1}$. The operator $E : \pi^* \Lambda^1 M \longrightarrow \pi^* \Lambda^k M$ can be expressed in terms of the connection form $A \in \Lambda^1 M \otimes \operatorname{End}(TM)$ as follows: at $(\lambda, m) \in X$ one has

$$\begin{aligned} E|_{(\lambda,m)}(v)(v_{i_1}, \dots, v_{i_k}) &= \lambda\left(A_v(v_{i_1}), v_{i_2}, \dots, v_{i_k}\right) + \lambda\left(v_{i_1}, A_v(v_{i_2}), \dots, v_{i_k}\right) \\ &\quad + \dots + \lambda\left(v_{i_1}, v_{i_2}, \dots, A_v(v_{i_k})\right), \end{aligned} \tag{2.5}$$

where A_v denotes $A(v)$. Comparing (2.5) and (2.4), we obtain

$$\Xi\left(v'_{i_1}, v'_{i_2}, \dots, v'_{i_{k+1}}\right)\Big|_{(\lambda,m)} = \frac{1}{(k+1)!}\sum (-1)^{|\sigma|} \lambda\left(A_{v_{\sigma(i_1)}}(v_{\sigma(i_2)}), v_{\sigma(i_3)}, \dots, v_{\sigma(i_{k+1})}\right). \tag{2.6}$$

Since ∇ and ∇_0 are torsion-free, one has $A_{v_{\sigma(i_1)}}(v_{\sigma(i_2)}) = A_{v_{\sigma(i_2)}}(v_{\sigma(i_1)})$. Therefore, the alternating sum (2.6) vanishes. $\square$

Remark 2.2. This argument repeats a more familiar argument showing that the restriction of the Hamiltonian symplectic form $\omega = \sum_i dp_i \wedge dq_i$ on T^*M to a horizontal space of a torsion-free connection is always zero.

3. The G_2-action on $\Lambda^2(\mathbb{R}^7)$ and $SU(3)$-action on $\Lambda^2(\mathbb{R}^6)$

3.1. Octonion algebra and quaternions

Let $V = \mathbb{R}^7$ be a 7-dimensional space equipped with a non-degenerate, non-split 3-form ρ inducing a G_2-action on V. Then V is equipped with the vector product, defined as follows: $x \star y = \rho(x, y, \cdot)^\sharp$. Here $\rho(x, y, \cdot)$ is a 1-form obtained by contraction, and $\rho(x, y, \cdot)^\sharp$ its dual vector field. It is not hard to see that $(V, \star)$ becomes isomorphic to the imaginary part of the octonion algebra, with $\star$ corresponding to half of the commutant. In fact, this is one of a definitions of an octonion algebra. The whole octonion algebra is defined $\mathbb{O} := V \oplus \mathbb{R}$, with the product given by

$$(x, t)\left(y, t'\right) = \left(ty + t'x + x \star y, g(x, y) + tt'\right).$$

Here, x, y and $ty + t'x + x \star y$ are vectors in V, and $t, t', g(x, y) + tt' \in \mathbb{R}$.

Given two non-collinear vectors in V, they generate a quaternion subalgebra in octonions. When these two vectors satisfy $|v| = |v'| = 1$, $v \perp v'$, the standard basis I, J, K in imaginary quaternions can be given by a triple $v, v', v \star v' \in V$.

A 3-dimensional subspace $A \subset V$ is called **associative** if it is closed under the vector product. The set of associative subspaces is in bijective correspondence with the set of quaternionic subalgebras in octonions.

The following well-known lemma is used further on in this section.

Lemma 3.1. *Let (V, ρ) be a 7-dimensional space with a G_2-structure, $v \in V$ a unit vector, and $W = v^\perp \subset V$ its orthogonal complement, equipped with a Hermitian structure as explained in Remark 1.8. Consider another unit vector $v_1 \in V$, $v_1 \perp v$, and let $W_1 := v_1^\perp$, with its own complex structure. Denote by $\omega_W \in \Lambda^2 W$ the Hermitian form on W. Then the restriction $\omega_W|_{W_1}$ is of Hodge type $(2,0) + (0,2)$.*

Proof. Let $A \subset V$ be a 3-dimensional associative space generated by v and v_1, and $H = A^\perp$. The intersection $W \cap W_1$ is a 5-dimensional space generated by H and $v \star v_1$, where $\star$ denotes the vector product. The restriction $\omega_W|_{W_1}$ vanishes on $v \star v_1$, because the complex structure on W maps $v \star v_1$ to $-v_1$. Therefore, $\omega_W|_{W_1}$ is an image of $\omega_W|_H$ under the standard orthogonal embedding $\Lambda^2 H \longrightarrow \Lambda^2 W_1$.

Three complex structures v, v_1, $v \star v_1$ define a quaternionic structure on H, and the corresponding three Hermitian forms give a hyperkaehler triple of symplectic structures on H. Since $\omega_W|_H$ is a real part of the standard $(2,0)$-form on H associated with v_1, it is a $(2,0) + (0,2)$-form on W_1. $\square$

3.2. G_2-action and $SU(3)$-action

Let $V = \mathbb{R}^7$ be a 7-dimensional space equipped with a non-degenerate, non-split 3-form ρ inducing a G_2-action on V as above. The space $\Lambda^2 V$ is a 21-dimensional representation of G_2. The Lie algebra $\mathfrak{g}_2$ can be considered as a subspace in $\Lambda^2 V$, because one has an embedding $\mathfrak{g}_2 \hookrightarrow \mathfrak{so}(V) = \Lambda^2 V$. This gives a 14-dimensional G_2-invariant subspace in $\Lambda^2 V$. There is also a 7-dimensional subspace given by an embedding $V \longrightarrow \Lambda^2 V$, $v \mapsto \rho \lrcorner v$. Since $\Lambda^2 V$ is 21-dimensional, this gives a decomposition $\Lambda^2 V = \Lambda^2_7 V \oplus \Lambda^2_{14} V$ of $\Lambda^2 V$ onto irreducible 7-dimensional and 14-dimensional G_2-subrepresentations.

Further on in this paper, we shall need the following linear-algebraic result.

Proposition 3.2. *Let $V = \mathbb{R}^7$ be a 7-dimensional space equipped with a G_2-action, and $\Lambda^2 V = \Lambda^2_7 V \oplus \Lambda^2_{14} V$ the irreducible decomposition obtained as above. For any $v \in V$, consider the Hermitian structure on its orthogonal complement $v^\perp$ (Remark 1.8). Let $\alpha \in \Lambda^2 V$ be a 2-form. Then the following conditions are equivalent:*

(i) *α lies in $\Lambda^2_{14}V$.*
(ii) *For any non-zero $v \in V$, the restriction of α to $v^\perp$ is of type $(1,1)$ with respect to the complex structure on $v^\perp$, and orthogonal to the Hermitian form.*
(iii) *For any non-zero $v \in V$, the restriction of α to $v^\perp$ is of type $(1,1)$ with respect to the complex structure on $v^\perp$.*

Proof. Denote by $W \subset V$ the orthogonal complement to v, with the natural $SU(3)$-structure constructed as in Remark 1.8. After the standard identification between $\mathfrak{so}(W)$ and $\Lambda^2 W$, the algebraic condition (ii) is translated to $\alpha|_W \in \mathfrak{su}(W)$. The restriction map $\Lambda^2 V \longrightarrow \Lambda^2 W$ is dual to the standard embedding $\mathfrak{so}(W) \hookrightarrow \mathfrak{so}(V)$. Under this embedding, $\mathfrak{su}(W)$ is mapped to $\mathfrak{g}_2$ (Remark 1.8), hence the restriction $\Lambda^2_{14}(V)|_W$ lies in $\mathfrak{su}(W)$. We proved the implication (i) $\Rightarrow$ (ii).

To obtain the converse implication, take some $\alpha \in \Lambda^2 V$ satisfying assumptions of (ii); since V is odd-dimensional, α has an annulator $R \subset V$, which is at least one-dimensional. Let $v \in R$ be a non-zero vector. Then α, considered as an element of $\mathfrak{so}(V)$, preserves $W = v^\perp$ and acts trivially on v, hence it is an image of $\alpha_0 \in \Lambda^2(W)$ under the standard embedding $\mathfrak{so}(W) \hookrightarrow \mathfrak{so}(V)$. By our assumptions, $\alpha_0 \in \mathfrak{su}(W)$, hence α belongs to an image of the standard embedding $\mathfrak{su}(W) \hookrightarrow \mathfrak{g}_2 = \Lambda^2_{14}(V)$.

To finish the proof of Proposition 3.2, it suffices to prove that (iii) implies (i). Let $\alpha \in \Lambda^2 V$ be a 2-form satisfying (iii). Consider a codimension 1 subspace $W \subset V$ constructed as above, with α in the image of the standard orthogonal embedding $\Lambda^2 W \hookrightarrow \Lambda^2 V$. Consider a decomposition $\alpha = \alpha_0 + t\omega_W$ of $\alpha \in \mathfrak{u}(W) = \Lambda^{1,1}(W)$ onto its traceless part $\alpha_0 \in \mathfrak{su}(W)$ and the part $t\omega_W$ proportional to the Hermitian form. Since $\alpha_0 \in \mathfrak{su}(W)$, this form satisfies $\alpha_0 \in \Lambda^2_{14}(V)$, because $\mathfrak{su}(W)$ lies in $\mathfrak{g}_2$. To prove that $\alpha \in \Lambda^2_{14}(V)$ it would suffice to show that $t\omega_W = 0$. However, since α_0 lies in $\Lambda^2_{14}(V)$, it satisfies the assumptions of Proposition 3.2(ii), hence satisfies Proposition 3.2(iii). Therefore, the same is true for $t\omega_W = \alpha - \alpha_0$: the restriction of $t\omega_W$ to any $W' = (v')^\perp$ of codimension 1 in V is of type $(1,1)$ with respect to the standard complex structure on W'. This implies $t = 0$, because, as follows from Lemma 3.1, the restriction of ω_W to another 6-dimensional subspace W' is of type $(2,0) + (0,2)$. $\square$

4. Integrability of the twistor CR-structure

The following claim is well known from the standard textbooks on differential geometry.

Claim 4.1. *(See [11].) Let M be a Riemannian n-manifold, $S^{n-1}M$ the sphere bundle consisting of all unit tangent vectors, and $T_{\mathrm{hor}}S^{n-1}(M) \subset TS^{n-1}(M)$ the horizontal tangent bundle associated with the Levi-Civita connection. Denote by $R \in \mathrm{Sym}^2 \Lambda^2 M$ the curvature of the Levi-Civita connection on M. Given a 2-form $\alpha \in \Lambda^2 M$, consider α as an element in $\mathfrak{so}(TM) = \Lambda^2 M$, acting on $S^{n-1}(M)$ by infinitesimal automorphisms, and let $\alpha^\star \in T_{\mathrm{vert}}S^{n-1}M$ be the corresponding vector field. Let $X, Y \in T_{\mathrm{hor}}S^{n-1}(M)$ be horizontal vector fields. Then the vertical component of $[X,Y]$ is equal to $R(X,Y)^\star$.*

We are going to prove Theorem 1.10, which says that the bundle $B^{1,0} \subset T_{\mathrm{hor}}X \otimes \mathbb{C}$ on a G_2-manifold M is involutive, where $X = S^6 M$. The strategy of the proof is the following. We consider the Frobenius bracket $[B^{1,0}, B^{1,0}] \xrightarrow{\Psi} TX \otimes \mathbb{C}/B^{1,0}$. First, we project $\Psi(X,Y)$, $X, Y \in B^{1,0}$ to $T_{\mathrm{vert}}X$ along the horizontal component $T_{\mathrm{hor}}X \otimes \mathbb{C} \supset B^{1,0}$, showing that this projection vanishes. This implies that Ψ is a map from $\Lambda^2 B^{1,0}$ to $T_{\mathrm{hor}}X \otimes \mathbb{C}/B^{1,0}$:

$$[B^{1,0}, B^{1,0}] \xrightarrow{\Psi} T_{\mathrm{hor}}X \otimes \mathbb{C}/B^{1,0}.$$

Then we prove that the Frobenius bracket $[\theta^\perp, \theta^\perp] \longrightarrow TX/\theta^\perp$ vanishes at $B \subset TX$, where $\theta \in TX$ is a natural section of $T_{\mathrm{hor}}X$ constructed as in Definition 1.9, and $\theta^\perp$ is its orthogonal complement. Since $B = T_{\mathrm{hor}}X \cap \theta^\perp$, this implies that the Frobenius bracket $[B^{1,0}, B^{1,0}] \xrightarrow{\Psi} T_{\mathrm{hor}}X \otimes \mathbb{C}/B^{1,0}$ lands in $\theta^\perp \cap T_{\mathrm{hor}}X = B$. This brings us to the situation familiar from complex (and almost complex) geometry, where we have a bundle B with an endomorphism $I \in \operatorname{End} B$, $I^2 = -\mathrm{Id}_B$, and the commutator map $[B^{1,0}, B^{1,0}] \longrightarrow B \otimes \mathbb{C}$ determines whether I is integrable. To prove that

$$[B^{1,0}, B^{1,0}] \subset B^{1,0}, \tag{4.1}$$

we construct a closed $(0,3)$-form $\bar{\Omega}$ which vanishes at $B^{1,0}$ and is non-degenerate at $B^{0,1}$, and use this form to prove (4.1).

Proof of Theorem 1.10. Step 1. By Claim 4.1, the commutator bracket

$$[T_{\mathrm{hor}}X, T_{\mathrm{hor}}X] \xrightarrow{\Psi_1} TX/T_{\mathrm{hor}}X$$

is expressed through the curvature R of the Levi-Civita connection on M, as follows:

$$\Psi_1(x,y)|_{(z,m)\in X} = R(x,y,z).$$

Consider $R_z := R(\cdot,\cdot,z)$ as a TM-valued 2-form on M. If M is holonomy G_2-manifold, this form lies in $\Lambda^2_{14} \otimes TM$, because the curvature of a holonomy G_2 manifold lies in $\Lambda^2_{14} \otimes \mathfrak{g}_2$. Therefore, R_z of type $(1,1)$ on B, as follows from Proposition 3.2.

This implies that $[B^{1,0}, B^{1,0}] \subset T_{\mathrm{hor}}X \otimes \mathbb{C}$. Conversely, if $[B^{1,0}, B^{1,0}] \subset T_{\mathrm{hor}}X \otimes \mathbb{C}$, this means that R_z is of type $(1,1)$ on every 6-dimensional subspace $W \subset T_m M$, and Proposition 3.2 implies that $R \in \Lambda^2_{14} \otimes \mathrm{End}(TM)$.

This already proves that integrability of the CR-structure $B^{1,0} \subset TX \otimes \mathbb{C}$ implies that (M, ρ) is a holonomy G_2-manifold. The converse implication takes more work.

Step 2. To study the bundle $B = \theta^\perp \cap T_{\mathrm{hor}}X$, we identify $X = S^6TM$ with the total space of the unit bundle $S^6T^*M \subset \mathrm{Tot}(T^*M)$. This is done using the Riemannian metric on TM. On $\mathrm{Tot}(T^*M)$ there is a standard 1-form Θ, dual to θ (see Section 2). The zero-space of this form is identified with $\theta^\perp$, and the corresponding Frobenius bracket $[\theta^\perp, \theta^\perp] \longrightarrow TX/\theta^\perp$ is a 2-form which is proportional to the restriction $d\Theta|_{\theta^\perp}$. However, $d\Theta = \Xi$ is the standard symplectic form on T^*M, and it vanishes on the horizontal subbundle, as follows from Lemma 2.1. This implies that the bracket $[\theta^\perp, \theta^\perp] \longrightarrow TX/\theta^\perp$ vanishes on $B \subset \theta^\perp$. In Step 1, we have shown that $[B^{1,0}, B^{1,0}] \subset T_{\mathrm{hor}}X \otimes \mathbb{C}$; the above argument implies that

$$\left[B^{1,0}, B^{1,0}\right] \subset \left(T_{\mathrm{hor}}X \cap \theta^\perp\right) \otimes \mathbb{C} = B \otimes \mathbb{C}.$$

Step 3. The 6-dimensional bundle $B \subset T_{\mathrm{hor}}M$ by construction comes equipped with a natural $SU(3)$-structure. The corresponding $(3,0)$-form can be written down as follows (Remark 1.8):

$$\left(\pi^*\rho + \sqrt{-1}\left(\pi^*\rho^*\right)\rfloor\theta\right)\big|_B. \tag{4.2}$$

Here, $\pi : X \longrightarrow M$ is the standard projection, ρ and $\rho^* := *\rho$ are the fundamental 3-form and 4-form on M, and (4.2) is the standard equation expressing the standard $(3,0)$-form on a codimension 1 subspace $W \subset V = \mathbb{R}^7$ in terms of the fundamental 3-form and 4-form on V. Let $\Omega := \pi^*\rho + \sqrt{-1}(\pi^*\rho^*)\rfloor\theta$. In Step 3, we are going to prove that $(d\Omega)|_{T_{\mathrm{hor}}X} = 0$. Since ρ is closed, this would follow from $d((\pi^*\rho^*)\rfloor\theta)|_{T_{\mathrm{hor}}X} = 0$. Let $Z := \mathrm{Tot}(\Lambda^3M)$, and $\phi : X \longrightarrow Z$ be an embedding mapping $(v, m) \in X = S^6M$ to $(\rho^*\rfloor v, m)$. Clearly, $\pi^*(\rho^*) = \phi^*(\Theta)$, where Θ is a 3-form on $\mathrm{Tot}(\Lambda^3M)$ defined in Section 2. Then $d((\pi^*\rho^*)\rfloor\theta) = \phi^*(\Xi)$. However, the connection on Λ^3M is compatible with the embedding $\Lambda^1(M) \longrightarrow \Lambda^3M$, hence the image $\phi(T_{\mathrm{hor}}X)$ lies in the horizontal subspace $T_{\mathrm{hor}}Z \subset TZ$. Therefore, to prove that

$$d\left(\left(\pi^*\rho^*\right)\rfloor\theta\right)\big|_{T_{\mathrm{hor}}X} = \phi^*(\Xi)|_{T_{\mathrm{hor}}X} = 0$$

it would suffice to show that $\Xi|_{T_{\mathrm{hor}}Z} = 0$. This follows from Lemma 2.1.

Step 4. Let $Z, T \in B^{0,1}$. Since Ω is of type $(3,0)$ on B, one has $\Omega\rfloor Z = \Omega\rfloor T = 0$. Therefore, Cartan's formula gives

$$d\Omega(X, Y, Z, T) = \Omega\left(X, Y, [Z, T]\right). \tag{4.3}$$

Taking a complex conjugate of the equation $[B^{1,0}, B^{1,0}] \subset B \otimes \mathbb{C}$ (Step 2), we obtain that $[Z, T] \in B \otimes \mathbb{C}$. Since $d\Omega|_B = 0$, (4.3) implies that the 2-form $\Omega([Z, T], \cdot, \cdot)$ is identically zero on B. However, Ω is a non-degenerate $(3,0)$-form on B, hence for any $\xi \in B \otimes \mathbb{C}$, the vanishing $\Omega(\xi, \cdot, \cdot) = 0$ implies that $\xi \in B^{0,1}$. We proved that $B^{0,1}$ is involutive. The same is true for $B^{1,0}$ by complex conjugation. □

References

[1] M. Berger, Sur les groupes d'holonomie homogene des varietes a connexion affine et des varietes Riemanniennes, Bull. Soc. Math. France 83 (1955) 279–330.
[2] A. Besse, Einstein Manifolds, Springer-Verlag, New York, 1987.
[3] Robert L. Bryant, Simon M. Salamon, On the construction of some complete metrics with exceptional holonomy, Duke Math. J. 58 (3) (1989) 829–850.
[4] J.L. Brylinski, Loop Spaces, Characteristic Classes and Geometric Quantization, Progr. Math., vol. 107, Birkhäuser, Boston, MA, 1993.
[5] Simon Donaldson, Ed Segal, Gauge theory in higher dimensions, II, arXiv:0902.3239.
[6] S.K. Donaldson, R.P. Thomas, Gauge theory in higher dimensions, in: The Geometric Universe (Oxford, 1996), Oxford Univ. Press, Oxford, 1998, pp. 31–47.
[7] M. Fernandez, A. Gray, Riemannian manifolds with structure group G_2, Ann. Mat. Pura Appl. (4) 132 (1982) 19–45 (1983).
[8] N. Hitchin, The geometry of three-forms in six and seven dimensions, J. Differential Geom. 55 (3) (2000) 547–576, math.DG/0010054.
[9] D. Joyce, Compact Riemannian 7-manifolds with holonomy G_2. I, II, J. Differential Geom. 43 (2) (1996) 291–328, 329–375.
[10] D. Joyce, Compact Manifolds with Special Holonomy, Oxford Math. Monogr., Oxford Univ. Press, Oxford, 2000.
[11] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, Wiley–Interscience, New York, 1969.
[12] Claude LeBrun, Twistor CR manifolds and three-dimensional conformal geometry, Trans. Amer. Math. Soc. 284 (1984) 601–616.
[13] László Lempert, Loop spaces as complex manifolds, J. Differential Geom. 38 (3) (1993) 519–543.
[14] T. Tao, G. Tian, A singularity removal theorem for Yang–Mills fields in higher dimensions, J. Amer. Math. Soc. 17 (3) (2004) 557–593, math.DG/0209352.
[15] G. Tian, Gauge theory and calibrated geometry, I, Ann. of Math. (2) 151 (1) (2000) 193–268, math.DG/0010015, 76 pp.
[16] M. Verbitsky, A formally Kaehler structure on a knot space of a G_2-manifold, arXiv:1003.3174, 24 pp.